A Note on Superfields and Noncommutative Geometry

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Abstract

We consider the supersymmetric field theories on the noncommutative $\mathbf{R}^4$ using the superspace formalism on the commutative space. The terms depending on the parameter of the noncommutativity Θ are regarded as the interactions. In this way we construct the $N = 1$ supersymmetric action for the $U(N)$ vector multiplets and chiral multiplets of the fundamental, anti-fundamental and adjoint representations of the gauge group. The action for vector multiplets of the products gauge group and its bi-fundamental matters is also obtained. We discuss the problem of the derivative terms of the auxiliary fields.

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In the past few years there has been much development in our understanding of the dynamics of supersymmetric gauge theories and superstring theories. Among these it has been discovered that the noncommutative gauge theories naturally appear [1, 2] when the D-branes with constant B fields is considered.

Recently Seiberg and Witten have argued that the noncommutative gauge theories realized as effective theories on D-branes are equivalent to some ordinary gauge theories [3]. In a single D-brane case, they have shown that the effective action for the D-brane is consistent with the equivalence if all derivative terms are neglected. Furthermore, it has been shown that the D-brane action, including derivative terms, computed in the string theory is consistent with the equivalence if we keep the two derivative terms but neglect the fourth and higher order derivative terms [4, 5, 6].

For deeper understanding for these phenomena, it is natural to investigate the field theories on the noncommutative geometry by the field theoretical approaches. In particular by the perturbative analysis it was found in [7] that the IR effects and UV effects are mixed in the noncommutative field theory.

To proceed further it may be important to study the noncommutative field theories with supersymmetry since their actions are highly constrained and we may understand the dynamics of these theories. To obtain the supersymmetric action, superfields on the noncommutative geometry may be desired. Explicit two-dimensional $N = 1$ noncommutative superspace was obtained in [8].

However, in this note, instead of investigating the noncommutative superspace formalism, we consider the ordinary superspace and superfields [9] and represent the noncommutative field theory using these notions. From the commutative supersymmetric action written by the superfields, we can obtain the noncommutative supersymmetric action by replacing the ordinary product between superfields to the $*$ product defined by the formula

$$f(x) * g(x) = e^{\frac{i}{2}\Theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}} f(x+\xi)g(x+\zeta)\Big|_{\xi=\zeta=0}. \tag{1}$$

Here we regard the additional terms depend on Θ^{ij} as the interaction terms with derivatives although we do not expand the $*$ product explicitly. We can do so because if Φ is superfield then $\partial^n\Phi$ is also superfields. Thus it is obvious that this action has the super-

symmetry and the R symmetry when it exists in the action with $\Theta = 0$ because $\partial_\mu \theta = 0$ where θ is the fermionic coordinate. This treatment of the superfields is similar to the notions of superspace and superfields in noncommutative geometry [10], in which only the chiral superfields has been considered.

In this paper, we consider the $N = 1$ supersymmetric theories on the noncommutative $\mathbf{R}^4$ although the above observation does not depend on the dimension of the spacetime and the number of the supersymmetries. We construct the $N = 1$ supersymmetric action for the $U(N)$ vector multiplets and chiral multiplets of the fundamental, anti-fundamental and adjoint representations of the gauge group. The actions for gauge fields of the products gauge groups and its bi-fundamental matters are also obtained.

It is argued that even if we do not require the supersymmetry, only these gauge groups and the matters are possible for the noncommutative gauge theories. We also find that the scalar potentials have some characteristic forms and discuss the problem of the derivative terms of the auxiliary fields.

The convention and notation taken in this paper are same as in [11] except for the spacetime indices, which are denote by $\mu, \nu, \cdots$ in this paper, and the gauge field A_μ.

First we consider the chiral superfields which satisfy $\bar{D}_{\dot{\alpha}} \Phi = 0$. Using the coordinate $y^m = x^m + i\theta\sigma^m\bar{\theta}$, these are written as $\Phi(y, \theta, \bar{\theta}) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$. The supersymmetry transformations are identical for commutative counterparts

$$
\begin{aligned}
\delta_\xi A &= \sqrt{2}\xi\psi, \\
\delta_\xi \psi &= i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi F, \\
\delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m\psi,
\end{aligned}
\tag{2}
$$

because we simply consider the ordinary superfields.

Defining

$$
(\prod_{i=1}^{n} f_i)_* = f_1 * f_2 * \cdots * f_n,
\tag{3}
$$

the most generic action which can be constructed from the chiral superfields Φ^i takes the form

$$
S = \int d^4x \left(\int d^2\theta d^2\bar{\theta} K(\Phi^i, \Phi^{\dagger j})_* + \left[\int d^2\theta W(\Phi^i)_* + h.c. \right] \right),
\tag{4}
$$

where $\int d^2\theta\,\theta^2 = 1$ and $\int d^2\bar\theta\,\bar\theta^2 = 1$. This is invariant under $K(\Phi^i, \Phi^{\dagger j})_* \to K(\Phi^i, \Phi^{\dagger j})_* + F(\Phi)_* + F(\Phi^\dagger)_*^\dagger$. Although the action of the component fields can be obtained straightforwardly, we will only give some examples below.

First we consider the action with $K = \Phi^\dagger * \Phi + a\Phi * \Phi * (\Phi^\dagger) + a^\dagger \Phi * (\Phi^\dagger) * (\Phi^\dagger)$ and $W = 0$, where a is some numerical constant. Note that

$$(A * B)^\dagger = B^\dagger * A^\dagger. \tag{5}$$

The part of the action which depends on F becomes

$$
\begin{aligned}
S|_F &= \int d^4x \left(F^\dagger F + (aA * F * F^\dagger + aF * A * F^\dagger + aF * F * A^\dagger + h.c) \right) \\
&= \int d^4x \left(F^\dagger F + (aF(F^\dagger * A) + aF(A * F^\dagger) + aF(F * A^\dagger) + h.c) \right).
\end{aligned} \tag{6}
$$

Here we have used that

$$\int d^4x\, A * B = \int d^4 AB = \int d^4x\, B * A, \tag{7}$$

which implies that the integral of the product of fields with $*$ product are unchanged by the cyclic rotation of the fields. Thus the action clearly contains the derivative of the auxiliary field F and it is difficult to eliminate it from the action using the equation of motion. Moreover F may become the propagating field if the noncommutative parameter $\Theta^{0\mu} \neq 0$ for some μ. To avoid these problems, we only consider the canonical Kähler potential $K = \sum_i \Phi_i^\dagger * \Phi_i$ below.

With this K, the action with non vanishing superpotential does not have the derivative of F then F can be eliminated. This can be seen from the fact that the terms which depend on F in the superpotential are linear in F. For example, the F dependent parts of the action with $W = a\Phi^n$ become

$$\int d^4x \left(\frac{1}{2} F^\dagger F + a \sum_{i=1}^{n} F\, (A^{n-i})_* * (A^{i-1})_* + h.c \right). \tag{8}$$

Next we consider the noncommutative Wess-Zumino model [12] [13] [14]

$$S_{WZ} = \int d^4x \left(\int d^2\theta d^2\bar\theta\, \Phi_i^\dagger * \Phi_i + \left[\int d^2\theta \left(\frac{1}{2} m_{ij}\Phi_i * \Phi_j + \frac{1}{3} g_{ijk}\Phi_i * \Phi_j * \Phi_k + g_i\Phi_i \right) + h.c. \right] \right), \tag{9}$$

where the mass m_{ij} is symmetric in their indices, however, the coupling g_{ijk} is not necessarily symmetric. By trancing the procedure for the $\Theta = 0$ case, we can easily find that

$$
\begin{aligned}
S_{WZ} =& \int d^4x \left(-\partial_\mu A_i^\dagger \partial^\mu A_i + i\partial_\mu \psi_i^\dagger \bar{\sigma}^\mu \psi_i + F_i^\dagger F_i \right) \\
&+ \int d^4x \left[\frac{1}{3} g_{ijk} \left(F_i\, A_j*A_k + F_j\, A_k*A_i + F_k\, A_i*A_j - A_i\, \psi_j*\psi_k - A_j\, \psi_k*\psi_i - A_k\, \psi_i*\psi_j \right) \right. \\
&\left. + g_i F_i + m_{ij} \left(A_i F_j - \frac{1}{2}\psi_i\psi_j \right) + h.c \right].
\end{aligned}
\tag{10}
$$

The equation of motions of F_i is

$$
F_i^\dagger = g_i + m_{ij}A_j + \frac{1}{3}\left(g_{ijk} + g_{kij} + g_{jki} \right) A_j*A_k,
\tag{11}
$$

and the supersymmetry transformation becomes (2) with this F_i. We note that the typical scalar potential has the form $A^\dagger*A^\dagger*A*A$ and the notion of holomorphy is still valid at $\Theta \neq 0$.

Now we consider the vector superfields $V = V^\dagger$ [15, 16] ,

$$
\begin{aligned}
V(x,\theta,\bar{\theta}) =\ & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\
&+ \frac{i}{2}\theta\theta \left[M(x) + iN(x) \right] - \frac{i}{2}\bar{\theta}\bar{\theta} \left[M(x) - iN(x) \right] \\
&- \theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x) \right] \\
&- i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x) \right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\Box C(x) \right].
\end{aligned}
\tag{12}
$$

Since the $*$ product contain derivative, the ordinary gauge invariant action for the vector superfields can not be generalized to $\Theta \neq 0$ case. Then we should introduce noncommutative gauge field $A_\mu = T^a A_\mu^a$, where T^a is the matrix for a representation of the gauge group G and satisfies that $(T^a)^\dagger = T^a$ and $\mathrm{Tr}(T^a T^b) = k$.

Hereafter we briefly discuss the some properties of the noncommutative gauge field without requiring supersymmetry in order to prepare to treat the vector superfield. We assume that the noncommutative gauge transformation is the naive generalization of the ordinary Non-Abelian gauge transformation,

$$
A_\mu \rightarrow U*A_\mu*U^{-1} + iU*\partial_\mu U^{-1},
\tag{13}
$$

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where $U = (e^{i\lambda})_*$, $U^{-1} = (e^{-i\lambda})_* = U^\dagger$ and $\lambda = T^a\lambda^a = \lambda^\dagger$. The infinitesimal version of this is

$$
\begin{aligned}
\delta_\lambda A_\mu &= \partial_\mu\lambda + i\lambda * A_\mu - iA_\mu * \lambda \\
&= T^a(\partial_\mu\lambda^a) + \frac{i}{2}[T^a, T^b](\lambda^a * A_\mu^b + A_\mu^b * \lambda^a) + \frac{i}{2}\{T^a, T^b\}(\lambda^a * A_\mu^b - A_\mu^b * \lambda^a). \quad (14)
\end{aligned}
$$

From this if $\{T^a, T^b\}$ is not a linear combination of T^d for some a, b, the gauge transformation is not closed. Thus the noncommutative gauge transformation is consistent only for unitary group $G = U(N)$ or its direct product $G = \prod_a U(N_a)^{(a)}$. In addition to this restriction, we should take T^a as the matrix for the fundamental or anti-fundamental representation by the requirement of the closure of (14). However T^a and $\tilde{T}^a = -{}^tT$, which represent the fundamental and anti-fundamental representations respectively, give the different gauge transformations via (14). Then we take $(T^a)^i{}_j$ to be the matrix for the fundamental representation, $N \times N$ Hermitian matrix.[†]

We can see that the representation of the gauge group G of the matter is also restricted to fundamental ($\mathbf{N}$), anti-fundamental ($\bar{\mathbf{N}}$), adjoint ($\mathbf{N} \times \bar{\mathbf{N}}$) or bi-fundamental ($\mathbf{N} \times \bar{\mathbf{M}}$) from the consideration of the possible form of the covariant derivative. In [17] this restriction has been shown for $G = U(1)$ case, where $T^1 = k^{\frac{1}{2}}$. For the fundamental matter represented as the column vector $(\psi)^i = \psi^i$, the gauge transformation and the covariant derivative are given by $\psi \to U * \psi$ and

$$
\mathcal{D}_\mu\psi = \partial_\mu\psi - iA_\mu * \psi, \quad (15)
$$

respectively. We can easily check $\mathcal{D}_\mu\psi \to U * (\mathcal{D}_\mu\psi)$ under the gauge transformation. Noting $\mathcal{D}_\mu\psi^\dagger \equiv (\mathcal{D}_\mu\psi)^\dagger = \partial_\mu\psi + i\psi^\dagger * A_\mu$, the covariant derivative for the anti-fundamental matter $(\tilde{\psi})_i = \tilde{\psi}_i$, which is transformed as $\tilde{\psi} \to \tilde{\psi} * U^{-1}$, is

$$
\mathcal{D}_\mu\tilde{\psi} = \partial_\mu\tilde{\psi} + i\tilde{\psi} * A_\mu. \quad (16)
$$

The adjoint matter $(\psi_{adj})^j{}_i$, which is transformed as $\psi_{adj} \to U * \psi_{adj} * U^{-1}$, can have the covariant derivative

$$
\mathcal{D}_\mu\psi_{adj} = \partial_\mu\psi_{adj} - iA_\mu * \psi_{adj} + i\psi_{adj} * A_\mu. \quad (17)
$$

[†] Of course we can choose the matrix of the anti-fundamental representation instead of the one for fundamental representation.

This is also seen from the covariant derivative for the bi-fundamental matter $(\psi_{N\bar{M}})^j_i$, where $j = 1 \ldots N$ and $i = 1 \ldots M$, is

$$\mathcal{D}_\mu \psi_{N\bar{M}} = \partial_\mu \psi_{N\bar{M}} - iA^{(1)}_\mu * \psi_{N\bar{M}} + i\psi_{N\bar{M}} * A^{(2)}_\mu. \tag{18}$$

Here $A^{(1)}_\mu$ and $A^{(2)}_\mu$ are the gauge fields for $U(N)$ and $U(M)$ respectively and the gauge transformation for it is given by $\psi_{N\bar{M}} \to U^{(1)} * \psi_{N\bar{M}} * U^{(2)^{-1}}$.

The interaction terms are severely constrained by the gauge symmetry and the possible forms of the terms are the polynomials of $\tilde{\psi} * (\psi_{adj}{}^n)_* * \psi$, and $\text{Tr}(\psi_{adj}{}^n)_*$ for $G = U(N)$. For the product group case, there are other terms which are allowed by the symmetry.

On the basis of this observation, we return to consider the vector superfields $V = T^a V^a$. We define the noncommutative super gauge transformation as

$$(e^{-2V})_* \to (e^{-2V'})_* = (e^{-i\Lambda^\dagger})_* * (e^{-2V})_* * (e^{i\Lambda})_*. \tag{19}$$

The chiral superfield

$$W_\alpha = \frac{1}{8}\bar{D}\bar{D}\left((e^{2V})_* * D_\alpha(e^{-2V})_*\right), \tag{20}$$

is transformed as

$$W_\alpha \to W_\alpha' = (e^{-i\Lambda})_* * W_\alpha * (e^{i\Lambda})_*. \tag{21}$$

Because of

$$V' = V + i(\Lambda - \Lambda^\dagger) + \cdots, \tag{22}$$

we can choose the Wess-Zumino gauge in which C, χ, M, N are eliminated. In the Wess-Zumino gauge we see

$$W_\alpha(y) = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F_{\mu\nu}(y) + \theta^2 (\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha, \tag{23}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu * A_\nu + iA_\nu * A_\mu, \tag{24}$$

and

$$\mathcal{D}_\mu \bar{\lambda} = \partial_\mu \bar{\lambda} - iA_\mu * \bar{\lambda} + i\bar{\lambda} * A_\mu. \tag{25}$$

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Defining the complex gauge coupling $\tau = \frac{\tilde{\theta}}{2\pi} + \frac{4\pi i}{g^2}$, we obtain the action of the non-commutative supersymmetric $U(N)$ gauge field theory,[‡]

$$
\begin{aligned}
S_V &= \frac{1}{16\pi k} \int d^4x d\theta^2 \mathrm{Tr}\left(-i\tau W^\alpha * W_\alpha + h.c.\right) \\
&= \int d^4x \mathrm{Tr}\left(-\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} - \frac{i}{g^2}\lambda\sigma^\mu(\mathcal{D}_\mu\bar{\lambda}) + \frac{1}{2g^2}DD - \frac{\tilde{\theta}}{64\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\right).
\end{aligned} \tag{26}
$$

Note that the definitions of $F_{\mu\nu}$ and $\mathcal{D}_\mu$ depend on Θ.

Next we consider the chiral superfields coupled to the vector superfields. The gauge transformations for the fundamental, anti-fundamental, adjoint and bi-fundamental chiral superfields are given by

$$
\begin{aligned}
\Phi &\rightarrow (e^{-i\Lambda})_* * \Phi, \\
\tilde{\Phi} &\rightarrow \tilde{\Phi} * (e^{i\Lambda})_*, \\
\Phi_{adj} &\rightarrow (e^{-i\Lambda})_* * \Phi_{adj} * (e^{i\Lambda})_*, \\
\Phi_{N\bar{M}} &\rightarrow (e^{-i\Lambda^{(1)}})_* * \Phi_{N\bar{M}} * (e^{i\Lambda^{(2)}})_*,
\end{aligned} \tag{27}
$$

respectively.

We can see that the supersymmetric gauge invariant actions including kinetic terms are

$$
\begin{aligned}
S_\Phi &= \int d^4x d^2\theta d^2\bar{\theta}\left(\Phi^\dagger * (e^{-2V})_* * \Phi\right) \\
&= \int d^4x \left(-(\mathcal{D}_\mu A^\dagger)(\mathcal{D}^\mu A) - i\psi^\dagger\bar{\sigma}^\mu(\mathcal{D}_\mu\psi) - A^\dagger * D * A \right. \\
&\qquad\left. -i\sqrt{2}A^\dagger * \lambda * \psi + i\sqrt{2}\psi^\dagger * \lambda^\dagger * A + F^\dagger F\right), \\
S_{\tilde{\Phi}} &= \int d^4x d^2\theta d^2\bar{\theta}\left(\tilde{\Phi} * (e^{2V})_* * \tilde{\Phi}^\dagger\right) \\
&= \int d^4x \left(-(\mathcal{D}_\mu\tilde{A})(\mathcal{D}^\mu\tilde{A}^\dagger) - i\tilde{\psi}\sigma^\mu(\mathcal{D}_\mu\tilde{\psi}^\dagger) + \tilde{A} * D * \tilde{A}^\dagger \right. \\
&\qquad\left. -i\sqrt{2}\tilde{A} * \lambda^\dagger * \tilde{\psi}^\dagger + i\sqrt{2}\tilde{\psi} * \lambda * \tilde{A}^\dagger + \tilde{F}\tilde{F}^\dagger\right), \\
S_{\Phi_{adj}} &= \int d^4x d^2\theta d^2\bar{\theta}\frac{1}{k}\mathrm{Tr}\left((e^{2V})_* * \Phi^\dagger_{adj} * (e^{-2V})_* * \Phi_{adj}\right) \\
&= \int d^4x \frac{1}{k}\mathrm{Tr}\left(-(\mathcal{D}_\mu A^\dagger_{adj})(\mathcal{D}^\mu A_{adj}) - i\psi^\dagger_{adj}\bar{\sigma}^\mu(\mathcal{D}_\mu\psi_{adj})\right.
\end{aligned}
$$

[‡] Although it is straightforward to obtain the action for the case of general Kähler potential, we only consider here the canonical Kähler potential in order to avoid the problem with the derivative terms of the auxiliary fields.

$$-A_{adj}^\dagger * D * A_{adj} + A_{adj} * D * A_{adj}^\dagger$$
$$-i\sqrt{2}A_{adj}^\dagger * \lambda * \psi_{adj} + i\sqrt{2}\psi_{adj}^\dagger * \lambda^\dagger * A_{adj}$$
$$-i\sqrt{2}A_{adj} * \lambda^\dagger * \psi_{adj}^\dagger + i\sqrt{2}\psi_{adj} * \lambda * A_{adj}^\dagger + F_{adj}^\dagger F_{adj}\Big),$$

$$S_{\Phi_{N\bar M}} = \int d^4x d^2\theta d^2\bar\theta \text{Tr}\left((e^{2V^{(2)}})_* * \Phi_{N\bar M}^\dagger * (e^{-2V^{(1)}})_* * \Phi_{N\bar M}\right)$$

$$= \int d^4x \text{Tr}\left(-(\mathcal{D}_\mu A_{N\bar M}^\dagger)(\mathcal{D}^\mu A_{N\bar M}) - i\psi_{N\bar M}^\dagger \bar\sigma^\mu (\mathcal{D}_\mu \psi_{N\bar M})\right.$$
$$-A_{N\bar M}^\dagger * D^{(1)} * A_{N\bar M} + A_{N\bar M} * D^{(2)} * A_{N\bar M}^\dagger$$
$$-i\sqrt{2}A_{N\bar M}^\dagger * \lambda^{(1)} * \psi_{N\bar M} + i\sqrt{2}\psi_{N\bar M}^\dagger * (\lambda^{(1)})^\dagger * A_{N\bar M}$$
$$\left.-i\sqrt{2}A_{N\bar M} * (\lambda^{(2)})^\dagger * \psi_{N\bar M}^\dagger + i\sqrt{2}\psi_{N\bar M} * \lambda^{(2)} * A_{N\bar M}^\dagger + F_{N\bar M}^\dagger F_{N\bar M}\right). \quad (28)$$

In the $\Theta = 0$ case $\text{Tr}(e^{2V}\Phi_{adj}^\dagger e^{-2V}\Phi_{adj})/k$ is equivalent to

$$\sum_{a,b=1}^{N^2} \Phi_{adj}^a{}^\dagger e^{-2\sum_{c=1}^{N^2} V^c (T_{adj}^c)_{ab}} \Phi_{adj}^b, \quad (29)$$

where T_{adj}^c is the matrix of the adjoint representation and we have used $e^Y X e^{-Y} = X + [X, Y] + \frac{1}{2}[Y[Y, X]] + \cdots$. However the generalization of (29) to the $\Theta \neq 0$ case is not noncommutative gauge invariant. For the anti-fundamental chiral superfield the similar phenomena can be shown and in general we should use the matrix of the fundamental representation T^a only.

We note that there are no derivative terms of the auxiliary fields D in the actions (28) and typical scalar potential are the form of $A^\dagger A A^\dagger A$, which is different from the one from the superpotential. In [18] it has been shown that the noncommutative complex scalar field theory with the interaction $A^\dagger A A^\dagger A$ does not suffer from IR divergences at one-loop insertions level. It is also seen that the classical moduli space of vacua is unchanged by varying Θ.

Finally as in the commutative case we can obtain the transformation of the supersymmetry in the Wess-Zumino gauge

$$\delta_\xi A = \sqrt{2}\xi\psi,$$
$$\delta_\xi \psi = i\sqrt{2}\sigma^\mu \bar\xi (\mathcal{D}_\mu A) + \sqrt{2}\xi F,$$
$$\delta_\xi F = i\sqrt{2}\bar\xi \bar\sigma^\mu (\mathcal{D}_\mu \psi) - 2i\bar\xi \bar\lambda A,$$
$$\delta_\xi A_\mu = -i\bar\lambda \bar\sigma^\mu \xi + i\bar\xi \bar\sigma^\mu \lambda,$$

$$\delta_\xi \lambda = \sigma^{\mu\nu} \xi F_{\mu\nu} + i\xi D,$$

$$\delta_\xi D = -\xi \sigma^\mu (\mathcal{D}_\mu \bar{\lambda}) - (\mathcal{D}_\mu \lambda^{(a)}) \sigma^\mu \bar{\xi}. \tag{30}$$

This formula is valid for the chiral superfields of any representation of G.

The form of the noncommutative gauge invariant superpotential are constrained as stated for the component fields. Using (10), we can easily write down the action of the component fields for any superpotential which is renormalizable at $\Theta = 0$.

It is possible to generalize these considerations to the extended superspace. On the other hand, we can construct the action with the extended supersymmetry by the $N = 1$ superfields for commutative case. We can easily construct the action with non-vanishing Θ corresponding to the commutative action with the extended supersymmetry. Even at $\Theta \neq 0$, these action has the extended supersymmetry because of the existence of the R symmetry which rotates the generators of the supersymmetry. In fact the $U(N)$ noncommutative gauge theory with one adjoint chiral superfield, N_f fundamental and N_f anti-fundamental chiral superfields and $W = \sqrt{2} \sum_{i=1}^{N_f} \tilde{\Phi}_{(i)} * \Phi_{adj} * \Phi_{(i)}$ has $N = 2$ supersymmetry. We can also obtain the noncommutative $N = 4$ supersymmetric action with $W = \text{Tr}(\Phi_{adj}^{(1)} * (\Phi_{adj}^{(2)} * \Phi_{adj}^{(3)} - \Phi_{adj}^{(3)} * \Phi_{adj}^{(2)}))$, where $\Phi_{adj}^{(i)}$ are three adjoint chiral superfields.

The effective theories of the D-branes on the orbifold are the quiver gauge theories [19] or the elliptic models [20] which have bi-fundamental matter. Thus it is interesting that the action for the supersymmetric gauge theories with bi-fundamental matters can be constructed.

In this paper, we have considered the $N = 1$ supersymmetric theories on the non-commutative $\mathbf{R}^4$. We have constructed the $N = 1$ supersymmetric action for the $U(N)$ vector multiplets and chiral multiplets of the fundamental, anti-fundamental and adjoint representations of the gauge group. The actions for gauge fields of the products gauge groups and its bi-fundamental matters have also been obtained. We have been argued that only these gauge groups and the matters are possible for the noncommutative gauge theories. We have also found that the scalar potentials have some characteristic forms and discussed the problem of the derivative terms of the auxiliary fields.

It is interesting to generalize the results obtained in this paper to the nonlinearly

realized supersymmetry. This is important because the supersymmetric DBI action which is the effective theory on a D-brane has this symmetry [21] and has been used for the instanton in the D-brane with the B field [3, 22] which is related to the noncommutative instanton [23].

Note added:

As this article was being completed, we received the preprint [24] which substantially overlap the present work.

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